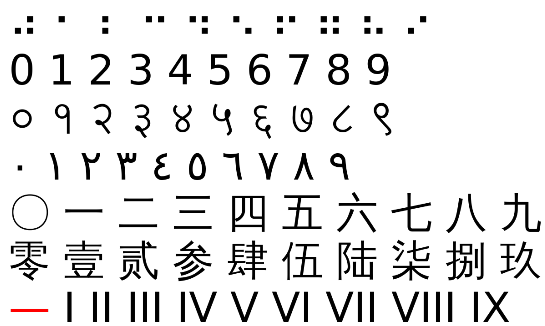
Grade 7/8 Math Circles
Week of 13th November
Types of Numbers

The History of Numbers

Numbers have been around since the dawn of time, from carving notches into bone for basic counting, to solving roots of polynomials. The earliest known record of a proper numbering system dates back 5000 to 6000 years ago in ancient Mesopotamia. Even earlier than that, archaeologists discovered the use of tally marks in wood, bone, and stone, dating all almost 40000 years ago. Some time evolved, and money was in full use – from silver coins, to gold ingots. Money needed to be divided evenly amongst its users, and fractions were readily available. The modern day numbers you and I use in our everyday lives was spread throughout India, and were called the *Arabic number system*.



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In the image above, we see the numerous numeral systems throughout history. Mathematicians take pride in their numbers; studying and classifying them extensively. They use numbers to count, measure, and label. Numbers are a very important aspect to mathematics as they quantify much of what we wish to describe; coefficients of polynomials, infinite series, evaluation of integrals, and so on.

Before moving on!

The term "set" will be used often throughout this lesson. A **set** is an unordered collection of distinct things, objects, or really anything you can fit between two curly brackets $\{\}$. For us, we'll use sets to put numbers in them, like so; $\{1, 2, 3\}$. We'll talk more about sets later on!

Natural numbers: $\mathbb{N}$

The natural numbers are the first type of numbers we'll examine. Natural numbers are called the counting numbers, they comprise of the numbers

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

The funny letter $\mathbb{N}$ is the symbol common used to denote the natural numbers, and recall that the the curly brackets $\{\}$ are used to denote a set. We call the set above, the *set of natural numbers*. The natural numbers do not include fractions, decimals, negative numbers, or zero (although some mathematicians argue $\mathbb{N}$ does contain 0). For example, we have that 100, 57, and 1029379426374273841 are natural numbers. On the other hand, $\frac{1}{2}$, π, and -1 are not!

Integers: $\mathbb{Z}$

Negative numbers were used before the concept of 0. In terms of accounting, many different cultures around the world used negative numbers to denote debt, or to cancel out transactions. The integers closely resemble the natural numbers, in so much as, they contain the natural numbers, and then all the negative numbers. They comprise of the numbers

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

Some examples of integers are -3, 0, and 10. On the other hand, $-\frac{1}{2}$, 1.5, and π are not! It's very interesting to note, as stated previously, the integers contain the natural numbers. Mathematically, we denote this as $\mathbb{N} \subset \mathbb{Z}$. We say, the "Natural numbers are a subset of the Integers".

Rational Numbers: $\mathbb{Q}$

What if we wished to represent numbers in between whole numbers, say between 0 and 1? With the use of fractions! Fractions add in a whole new set of numbers, called the **rational numbers**. We call them this, because they are in reference to numbers that can be represented as the ratio of two integers; that is

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z}, q \neq 0 \right\}$$

What you see above is the set notation for the set of rational numbers.

$$\left\{ \underbrace{\frac{p}{q}}_{\text{Set contains ratios of } \frac{p}{q}} \,\middle|\, \overbrace{p, q \in}^{p \text{ and } q \text{ are integers}} \mathbb{Z},\ \underbrace{q \neq 0}_{q \text{ is not zero, you cannot divide by } 0} \right\}$$

Note that the symbol "$\in$" just means "an element of". We denote the rational number with $\mathbb{Q}$, as it stands for *quotient*. Rational numbers date back to the Indians in 458, when they used decimals in their works regarding astronomy. Since we note that any integer n can be expressed as a fraction, namely as $\frac{n}{1}$, we have that the integers are a subset of the rationals. Therefore, the chain continues as follows $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

Irrational Numbers: $\bar{\mathbb{Q}}$

It was long believed by the ancient Greeks, namely Pythagoras, that all numbers were rational. Though, after a clever proof by a man named Hippasus, their entire world was flipped upside down when he showed that there exists number that cannot be represented as a fraction of two numbers.

Stop and Think

Input $\sqrt{2}$ into your calculator, what is the output? Can you try representing this as a fraction?

Here is a quick proof that $\sqrt{2}$ is irrational. Assume for contradiction that $\sqrt{2}$ is rational, that means

$$\sqrt{2} = \frac{p}{q} \quad \text{where } p \text{ and } q \text{ are integers}$$

and we can assume that p and q have no common factors. Following along with this logic, we'd have that

$$2 = \frac{p^2}{q^2} \implies 2q^2 = p^2$$

This means that p^2 is even, which means that p is even (since when you multiply by even number by itself, an even number, you get an even number). If p is even, so is q since they are related by the expression $p^2 = 2q^2$. Thus, both p and q are even, and therefore both share a common factor of 2, contradicting our initial statement that they have no common factors. Thus, the square root of 2 cannot be rational, it cannot be expressed as a ratio of two integers.

Irrational numbers are numbers which don't have repeated decimals, they are never ending. It's very important to note though that numbers with repeating decimals that are never ending, such as $\frac{1}{3} = 0.3333...$ are rational, since it's the same number repeated. For an irrational number, the never ending sequence of decimals has to be **random**.

It's very important to note that the square root of any number that ISN'T a perfect square, is irrational! That means, for example, $\sqrt{5}$ and $\sqrt{6}$ are irrational, but $\sqrt{4}$ and $\sqrt{16}$ are rational.

Real Numbers: $\mathbb{R}$

We now wish to encompass all the previously mentioned numbers, we do this with the set of real numbers, $\mathbb{R}$. The real numbers are defined to be any number that could ever exist on the number line, it includes the naturals, integers, rationals, and irrationals. Our chains grows even more, with $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, as well as $\bar{\mathbb{Q}} \subset \mathbb{R}$. Therefore, some real numbers are π, $\sqrt{2}$, 10, $\frac{1}{2}$, and -2.

Before moving on!

The reason the irrationals are denoted as $\bar{\mathbb{Q}}$ is because it is the **complement** of the rationals, which is denoted with a bar atop the set. The complement of a set is everything else that could possible exist outside of the set. We'll talk more about this in detail later in the lesson. Since the rationals is the set of all real numbers numbers that can be represented as a fraction of two integers, the complement is all the real numbers that cannot be represented as a fraction of two integers, the *irrationals*!

Exercise 1

Determine which sets the following numbers belong to. Make sure to list them all, even if they are subsets of another set already given!

a) 11	*b*) $\sqrt{4}$
c) $-\dfrac{2}{1}$	*d*) $\sqrt{11}$
e) 4.333...	*f*) π

Complex and Imaginary Numbers: $\mathbb{C}$ & $\mathbb{I}$

Let's try solving the following equation for x. What do you notice?

$$x^2 + 1 = 0$$

We notice that we are looking for a number that when squared, is equal to -1. This number doesn't exist in $\mathbb{R}$, it exists in $\mathbb{I}$. We call it i. That is $i^2 = -1$.

$$(i)^2 + 1 = (-1) + 1 = 0$$

We call these numbers imaginary numbers.

Example 1
Consider the equation $4x^2 + 2 = -7$, we have that

$$\begin{aligned} 4x^2 &= -9 \\ x^2 &= -\frac{9}{4} \\ \sqrt{x^2} &= \sqrt{-\frac{9}{4}} \\ x &= \pm\frac{3}{2}i \end{aligned}$$

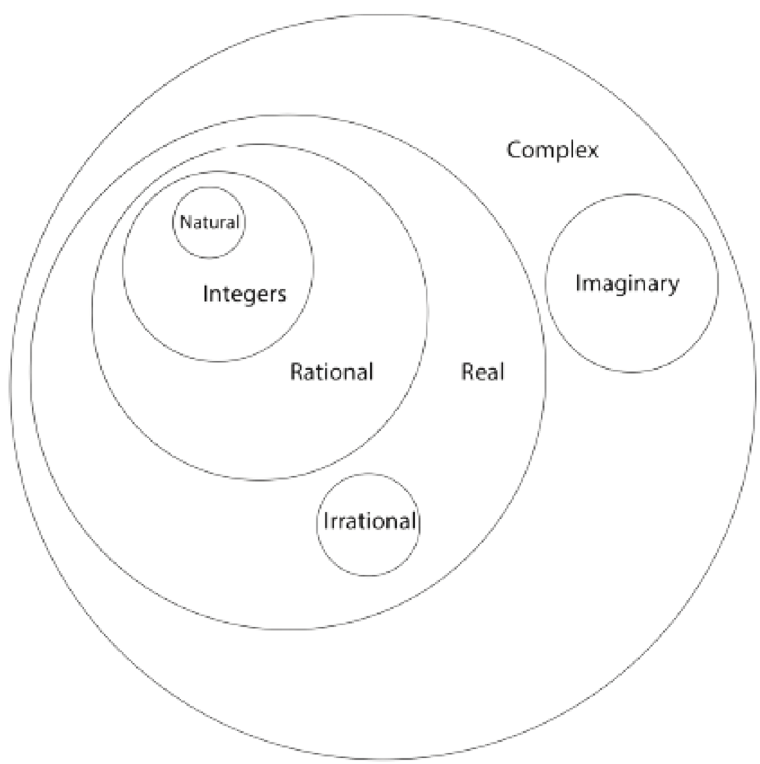
Exercise 2
Find what imaginary numbers satisfy the following equations. Write it as $x = ai$

$$x^2 + 6 = -3 \qquad 2x^2 + 8 = 0 \qquad x^2 + 5 = 0$$

Complex numbers are very closely related, they are numbers of the form $a + bi$, where $a, b \in \mathbb{R}$. They combine the imaginary numbers, with the real numbers. We denote a to be the real part, and b the imaginary part. Some examples include $1+i$, $2+3i$, and $\frac{1}{\sqrt{2}}+\pi i$. We generally use the following notation: let's say we have some complex number, $z = a + bi$, then $\text{Re}(z) = a$, and $\text{Im}(z) = b$. Since an imaginary number is just a complex number with real part equal to 0, i.e $a = 0$, we have that $\mathbb{I} \subset \mathbb{C}$. And since a real number is just a complex number with imaginary component equal to 0, i.e $b = 0$, we have that $\mathbb{R} \subset \mathbb{C}$. Thus, we've finally completed the chain of numbers,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \qquad \bar{\mathbb{Q}} \subset \mathbb{R} \qquad \mathbb{I} \subset \mathbb{C}$$

We can visualize this with a large Venn diagram!



Where do Complex Numbers Live?

All the real numbers lie on the real number line, which looks like

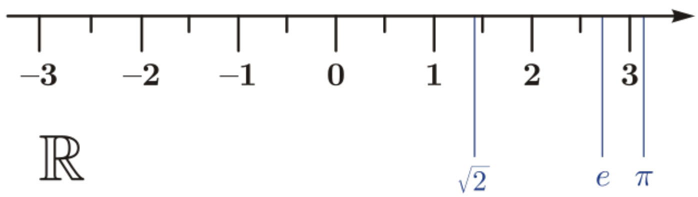
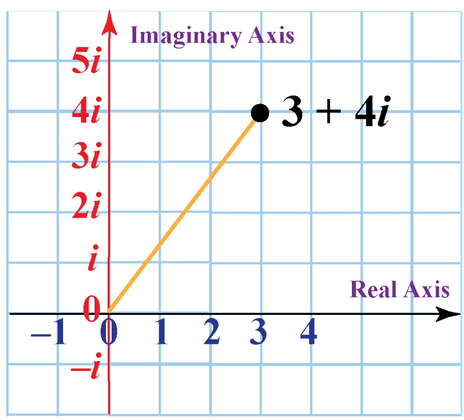


Image retrieved from Wikipedia

They are all aligned on a straight axis. Since the complex number live in a whole new world, what we do is extend the number line vertically!



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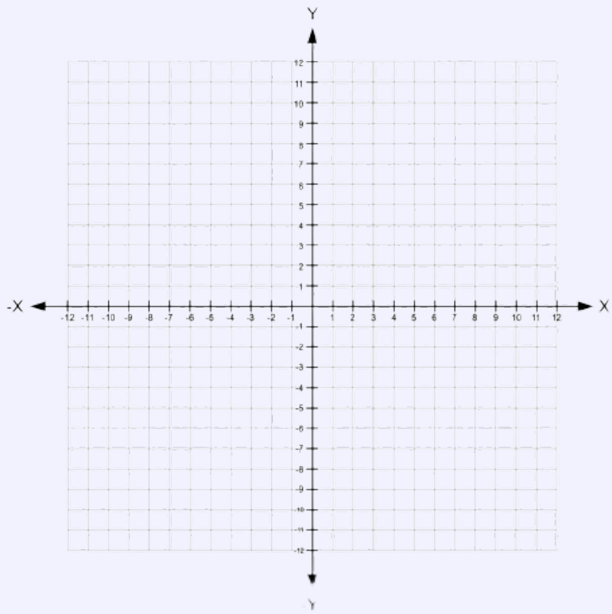
Thus, if we have a complex number, we plot the imaginary part on the y-axis, and the real part along the x-axis.

Exercise 3
Plot the following complex numbers on the graph

$$a)\ 2i \quad b)\ 3+3i \quad c)\ -1+i \quad d)\ 9-\tfrac{2}{3}i$$



How do Complex numbers work?

Addition and subtraction, and multiplication and division are operations that are valid in all types of numbers. For example

- $\mathbb{N}, \mathbb{Z}, \bar{\mathbb{Q}}, \mathbb{R} : a \cdot b = a \times b$
- $\mathbb{Q} : \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$

But what about complex numbers? Say we have two complex numbers z and w, what would adding and subtracting them look like, what would multiplying or dividing them look like? We define addition and subtraction as follows. Consider $z = a + bi$ and $w = c + di$

$$\begin{aligned} z + w &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \end{aligned} \qquad \begin{aligned} z - w &= (a + bi) + (c + di) \\ &= (a - c) + (b - d)i \end{aligned}$$

Example 2
Consider $z = 1 + 7i$ and $w = 4 + 3i$, then

$$\begin{aligned} z + w &= (1 + 7i) + (4 + 3i) \\ &= (1 + 4) + (7 + 3)i \\ &= 5 + 10i \end{aligned} \qquad \begin{aligned} z - w &= (1 + 7i) - (4 + 3i) \\ &= (1 - 4) + (7 - 3)i \\ &= -3 + 5i \end{aligned}$$

So, when we add two complex numbers, we effectively add their real parts together, and the imaginary parts together respectively. For subtraction, subtract their real parts, and subtract their imaginary parts together. Note though, that we're always left with another number of the form $x + yi \in \mathbb{C}$.

Exercise 4
Compute the following expressions

a) $(1 + 6i) + (3 + 4i) =$

b) $(4 + 2i) - (8 - 3i) =$

c) $(3 + 2i) + ((2 - 1i) + (3 - 2i)) =$

We'll do the same for multiplication, and afterwards for division. For multiplication, we define it as

$$\begin{aligned} z \cdot w &= (a + bi) \cdot (c + di) \\ &= (ac - bd) + (ad + cb)i \end{aligned}$$

The reason it's defined as such, is because we can treat the numbers as binomials, and expand like we would non-complex binomials, like so

$$\begin{aligned} z \cdot w &= (a + bi) \cdot (c + di) \\ &= ac + adi + cbi + bdi^2 \\ &= ac + adi + cbi - bd \\ &= (ac - bd) + (ad + cb)i \end{aligned}$$

Example 3
Consider $z = 1 + 2i$ and $w = 2 + 3i$, then

$$\begin{aligned} z \cdot w &= (1 + 2i) \cdot (2 + 3i) \\ &= (1 \cdot 2 - 2 \cdot 3) + (1 \cdot 3 + 2 \cdot 2)i \\ &= (2 - 6) + (3 + 4)i \\ &= -4 + 7i \end{aligned}$$

For division, we define it as follows

$$\begin{aligned} \frac{z}{w} &= \frac{a + bi}{c + di} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{cb - ab}{c^2 + d^2} i \end{aligned}$$

To understand where this comes from, we want to introduce a fairly new concept, the fact that when we have a fraction, comprised of two complex numbers, we want to simplify the expression so that there isn't an i in the denominator. We'll multiply the fraction by the conjugate. The conjugate of a complex number, z, is denoted $\bar{z}$.

$$\text{if} \quad z = a + bi \quad \text{then} \quad \bar{z} = a - bi$$

All we do is flip the sign of the imaginary part! So, when dividing complex numbers, we employ the conjugate as follow

$$\begin{aligned} \frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{(a + bi) \cdot (c - di)}{(c + di) \cdot (c - di)} \\ &= \frac{ac - adi + cbi - bdi^2}{c^2 - cdi + cdi - d^2 i^2} \\ &= \frac{(ac + bd) + (cb - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{cb - ab}{c^2 + d^2} i \end{aligned}$$

Example 4
Consider $z = 1 + 2i$ and $w = 4 - 2i$, then

$$\begin{aligned}\frac{z}{w} &= \frac{1+2i}{4-2i} \\ &= \frac{1\cdot 4 + 2\cdot(-2)}{4^2+(-2)^2} + \frac{4\cdot 2 - 1\cdot 2}{4^2+(-2)^2}i \\ &= \frac{4-4}{16+4} + \frac{8-2}{16+4}i \\ &= \frac{0}{20} + \frac{6}{20}i \\ &= \frac{3}{10}i\end{aligned}$$

Exercise 4
Simplify the following expressions

a) $\dfrac{1+i}{1-i}$

b) $(2-3i)\cdot\left(\dfrac{1}{2} - \dfrac{1}{3}i\right)$

c) $((4+1i)-(2+2i))\cdot(1-i)$

d) $(2-3i)\cdot((3-i)+(2+2i))$

e) $\dfrac{1-3i}{5-2i}$

f) $\left(\dfrac{3}{4}i\right)\cdot\left(\dfrac{4}{3}i + \dfrac{1}{3}\right)$

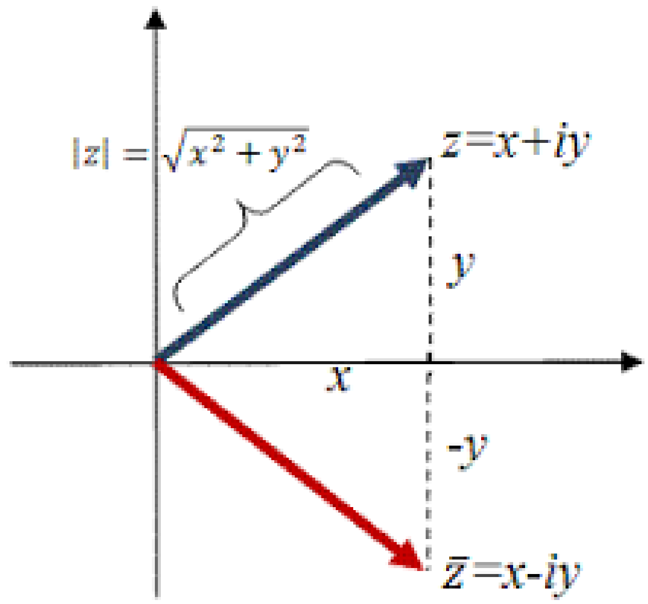
Something else that mathematicians care about is the **modulus** of a complex number. We also call this the **magnitude**. Given any complex number $z = a + bi$, the modulus is defined as

$$|z| = \sqrt{a^2+b^2}$$

Example 5
Consider $z = 3 + 4i$, then

$$|z| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

If we pay close attention to the output, it looks awfully similar to a very familiar theorem, the Pythagorean Theorem! So, the modulus gives us the length of the line segment from the graphed complex number.



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We require this tool because complex numbers are not ordered, you cannot say one complex number is bigger than another (i.e you cannot say $z > w$). Since the absolute value denotes the distance from 0, likewise, the modulus gives us the Euclidean distance from 0 in the complex plane.

Exercise 5
Find the modulus of the following complex numbers.

$$a)\ z = 1 - i \qquad b)\ z = \sqrt{2} + \sqrt{2}i \qquad c)\ z = 1 + i \qquad d)\ z = \frac{-1-i}{1-i}$$

Sets

All these numbers live inside of sets, but how do these sets work? There are unique operations with sets, just like with numbers. When we had two sets, we can "add" them just like numbers, we call this the **union** of two sets. This is denoted with the symbol $\cup$. Say we have two sets, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, the union of these two sets is

$$A \cup B = \{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$$

Say we have two sets, and we want to know what elements are shared in **both** sets, we take the **intersection** of the two sets. This is denoted with the symbol $\cap$. Say we have two sets, $A = \{1, 2, 3\}$ and $B = \{1, 3, 4\}$, the intersection of these two sets is

$$A \cap B = \{1, 2, 3\} \cap \{1, 3, 4\} = \{1, 3\}$$

We say that $A \cap B = \varnothing$ if the two sets share no elements in common ($\varnothing$ is called the empty set). Our last notable set notation is the **complement**. The complement of a set is everything that isn't inside of the set itself. Say the only numbers that existed were $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If I have a set $A = \{0, 1, 2, 3\}$, the complement, denoted $\bar{A}$, is $\bar{A} = \{4, 5, 6, 7, 8, 9\}$. This is why we call the irrational numbers $\bar{\mathbb{Q}}$! Because the irrational numbers is the set of all real numbers that **cannot** be written as a fraction! Let's look at numbers in their sets once more. Recall that

$$\begin{aligned}
&\mathbb{N} = \{1, 2, 3, 4, ...\} \\
&\mathbb{Z} = \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\} \\
&\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z}, q \neq 0 \right\}
\end{aligned}$$

Exercise 6
Evaluate the following set expressions

$a)\ \mathbb{N} \cap \mathbb{I}$	$b)\ \mathbb{N} \cup \mathbb{Z}$
$c)\ \mathbb{Q} \cup \overline{\mathbb{Q}}$	$d)\ \mathbb{R} \cap \mathbb{I}$
$e)\ \mathbb{N} \cap \mathbb{Z}$	$f)\ \mathbb{R} \cup \mathbb{I}$